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Detecting Nonergodicity in Continuous-Time Spin Systems

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We develop an empirical test aimed at detecting nonergodicity from a single sample of a spin system. We show that the test is asymptotically correct, and we give explicit asymptotics for the error probability. The key tool consists in some new large-deviation estimates.

KEY WORDS: Ergodicity; large deviations; empirical test.

1. INTRODUCTION

In this paper we analyze some aspects of the behavior of stochastic systems that are comprised of a large number of particles. The structure of the processes we consider becomes particularly rich in the limit of infinitely many particles (thermodynamic limit); some of these systems, for instance, may lose ergodicity after having performed the thermodynamic limit, i.e., the limit process may have more than one invariant measure. It is well known that the knowledge of the nonergodicity of a system in its infinite-particle limit may also help to understand relevant features of finite (but large) systems, such as metastability.[1,2]

In general, if we are given an infinite-particle stochastic system, it is quite hard to establish its ergodic properties by knowing only the equation of the dynamics. For the spin systems we deal with in this paper there are conditions (indeed quite severe) that guarantee ergodicity,[8] but nonergodicity can be proved only for a few special models. The approach we propose here consists in detecting nonergodicity via an empirical test. We assume we can observe a finite but large portion of the system for a large amount of time; we then test the unique realization we have observed, and

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decide whether the system is ergodic or not. We also get asymptotics for the probability that the test gives the wrong answer.

Before giving a more accurate description of the model and of the test, we remark that both the test we present and the method to justify it are not new. They were introduced by Comets[3] as a way to detect phase transitions in Gibbs fields with short-range interaction. Our contribution in this paper mainly consists in showing that Comets' ideas can be applied to the setting of continuous-time spin systems. We have to use the machinery of space-time large deviations developed in ref. 4 and 5, but this turns out not to be enough. Nonergodic systems are in fact characterized by the existence of "moderate" fluctuations, i.e., fluctuations whose probability is much higher than that of "genuine" large deviations. Giving bounds for these moderate fluctuations turns out to be harder than in the context of Gibbs fields, and is the main technical achievement of this work.

We now give an informal description of the test for ergodicity, in the context of continuous-time spin systems. A spin-flip system in the infinite lattice is a Markov process with state space $\{-1, 1\}^{\mathbf{Z}^d}$, $d \geqslant 1$. If we denote by $\sigma = \{\sigma(i): i \in \mathbf{Z}^d\}$ an element of $\{-1, 1\}^{\mathbf{Z}^d}$, the transition probability of the spin-flip system is characterized by

$$\operatorname{Prob}\{\sigma_{t+\Delta t}(i) = -\sigma_t(i) \mid \sigma_t\} = c(\theta_i \sigma_t)\, \Delta t + o(\Delta t)$$

for every $i \in \mathbf{Z}^d$, where θ_i is the shift map in $\{-1, 1\}^{\mathbf{Z}^d}$ defined by $\theta_i \sigma(j) = \sigma(i+j)$. The positive function c is called the *flip rate*. In this paper we assume $c(\cdot)$ to be a strictly positive and *local* function, i.e., depending only on the spins $\{\sigma(i): i \in \Lambda\}$ with Λ a finite subset of $\mathbf{Z}^d$. We notice that just by replacing $\mathbf{Z}^d$ by the N-torus $\mathbf{Z}^d/N\mathbf{Z}^d$ we can define, in an analogous way, a *finite* spin system. Since we assume the flip rates to be strictly positive, it follows that the finite system is an ergodic Markov chain, while there are many examples of nonergodic spin-flip systems on $\{-1, 1\}^{\mathbf{Z}^d}$.

Just to fix ideas, assume we are dealing with a system having two translation-invariant, ergodic equilibrium measures p_+ and p_-, with $E^{p_+}(\sigma(0)) > 0$ and $E^{p_-}(\sigma(0)) < 0$, and that the system converges to p_+ (resp. p_-) if initialized in a configuration σ with $\lim_n (1/n) \sum_1^n \sigma(i) > 0$ (resp. <0). Of course there are configurations for which this limit is not defined, but this is unimportant for this informal discussion. Suppose we start from a configuration having all spins equal to $+1$, and that we observe the evolution of the finite but large subsystem $\{\sigma(i): |i| \leqslant n\}$. Due to the random noise, there is a high probability that, within a time of order $\leqslant n$, there is a square $S \subset \{|i| \leqslant n\}$, of volume $\log^d n$, such that the majority of spins in S is -1. Since p_- is an "attractor" for those configurations with most of the spins negative, there are good chances that S keeps its "negativity" for a certain amount of time until it "realizes" it belongs to a larger system attracted by p_+. This amount of time is presumably at least of order $\log n$, which is the time needed for the sites in the *core* of S to become sensibly influenced by the sites in S^c (due to the locality of the interaction). To get a global picture we could say that, although the system converges to p_+, there are small, but not too small, space-time islands with mostly negative spins.

The discussion above suggests the following strategy. Suppose we observe the evolution of the sites in a square of volume n^d from time $t=0$ to time $t=n$. Our observations are therefore relative to a space-time square of volume n^{d+1}. We now divide this square into smaller subsquares of side $K \log n$, $K>0$, to be properly chosen. We expect that there is at least one of these subsquares whose average spin is negative, while the average spin in most of the other squares is close to $E^{p_+}(\sigma(0))$; in other words, a sensible difference in the average spin of different space-time boxes is what allows us to recognize that the system is not ergodic.

Surprisingly enough, this idea works. In general the extremal invariant measures may be more than two, and may not be distinguishable by the average spin, but by more complicated functionals. In all cases, however, nonergodic systems are characterized by the fact that the empirical averages of *some* functionals over space-time boxes of appropriate size is, with high probability, sensibly varying. This allows one to design a test to answer the following question: given $\varepsilon>0$, are there two invariant measures for the system whose distance (e.g., in Prohorov metric) is greater that ε? We prove that our test is asymptotically correct, i.e., the probability of giving the wrong answer goes to zero as the size of the observed space-time window increases to infinity, and we analyze the asymptotics of this probability.

2. LARGE DEVIATIONS

In this section we define the continuous-time processes we will be dealing with and summarize the large-deviations results proved in refs. 4 and 5. The Markov processes we are going to define take value on $\{-1, 1\}^{\mathbf{Z}^d}$, i.e., for any site i of the d-dimensional lattice $\mathbf{Z}^d$ there is an associated spin value. The updating mechanism is specified by assigning a nonnegative function $c(i, \sigma)$, defined for $i \in \mathbf{Z}^d$, $\sigma \in \{-1, 1\}^{\mathbf{Z}^d}$. The probability of changing the sign of the spin at the site i during a time interval of length Δt, conditioned on the knowledge of the whole configuration at time t, is given by

$$P\{\sigma_{t+\Delta t}(i) = -\sigma_t(i) \mid \sigma_t\} = c(i, \sigma_t)\, \Delta t + o(\Delta t)$$

Moreover, spins at different sites are updated independently. In particular the probability of changing the spin at two different sites in the same interval $[t, t+\Delta t]$ is $o(\Delta t)$. The functions $\{c(i, \cdot): i \in \mathbf{Z}^d\}$ are usually called *flip rates*.

The informal definition we have just given can be made rigorous as follows. First of all we provide $\{-1, 1\}^{\mathbf{Z}^d}$ with the product of the discrete topology on X. The corresponding space of real continuous functions is denoted by $\mathscr{C}(\{-1, 1\}^{\mathbf{Z}^d})$; it becomes a Banach space with the usual sup-norm. We say that a function $f: \{-1, 1\}^{\mathbf{Z}^d} \to \mathbf{R}$ is local if its dependence on $\sigma \in \{-1, 1\}^{\mathbf{Z}^d}$ is only through $\{\sigma(i): i \in \Lambda\}$, where Λ is some finite subset of $\mathbf{Z}^d$. We denote by D the set of local functions. On D we can define the following operator:

$$L^c f(\sigma) = \sum_{i \in \mathbf{Z}^d} c(i, \sigma)[f(\sigma^i) - f(\sigma)]$$

where

$$\sigma^i(j) = (-1)^{\delta_{i,j}} \sigma(j)$$

It is proved in ref. 8 that, under the assumption

$$\sup_{i \in \mathbf{Z}^d} \sum_{j \in \mathbf{Z}^d} \sup_{\eta \in \{-1,1\}^{\mathbf{Z}^d}} |c(i, \eta) - c(i, \eta^j)| < \infty \tag{1}$$

the closure of L^c in $\mathscr{C}(\{-1, 1\}^{\mathbf{Z}^d})$ generates a Markov semigroup. Moreover, the corresponding Markov process is a Feller process. Notice that condition (1) essentially says that $c(i, \sigma)$ does not depend too much on the spin of sites that are far from i. We notice that (1) is satisfied when the flip rates are *translation invariant* [i.e., $c(i, \sigma) = c(0, \theta_i \sigma)$, where $\theta_i \sigma(j) = \sigma(i+j)$] and *local* [i.e., $c(0, \sigma)$ depends only on $\{\sigma(i): i \in \Lambda\}$, where Λ is a finite subset of $\mathbf{Z}^d$]. Only these types of models will be considered in the rest of the paper.

We now assume $c: \{-1, 1\}^{\mathbf{Z}^d} \to \mathbf{R}^+$ is a local and strictly positive function. As we have just seen, the operator

$$L^c f(\sigma) = \sum_{i \in \mathbf{Z}^d} c(\theta_i \sigma)[f(\sigma^i) - f(\sigma)]$$

is the generator of a Feller semigroup. We denote by $\{P^c_{0,\xi}: \xi \in \{-1, 1\}^{\mathbf{Z}^d}\}$ the corresponding family of conditional probability measures. In particular, for $c \equiv 1$, we write $P_{0,\xi}$ in place of $P^1_{0,\xi}$. Notice that $P_{0,\xi}$ is simply the product measure $\prod_{i \in \mathbf{Z}^d} P_{0,\xi(i)}$, $P_{0,\xi(i)}$ being the Markov family of a Poisson-spin process with intensity one. For obvious reasons the process generated by L^c with $c \equiv 1$ is called a *noninteracting* spin system. Let $\Omega = D(\mathbf{R}, \{-1, 1\}^{\mathbf{Z}^d})$ be the space of the cadlag functions from $\mathbf{R}$ to $\{-1, 1\}^{\mathbf{Z}^d}$, provided with the Skorohod topology; Ω is the path space for the process. On Ω we define the family of space-time shift maps $\{\theta_{t,n}: t \in \mathbf{R}, n \in \mathbf{Z}^d\}$ defined by

$$(\theta_{t,n}\omega)_s(i) = \omega_{s+t}(i+n)$$

Definition 2.1. A probability measure Q on Ω is said to be stationary if it is invariant for all the maps $\theta_{t,n}$.

We denote by $\mathcal{M}_s(\Omega)$ the set of stationary measures, provided with the weak topology. The Borel sets for this topology provide Ω with a structure of measurable space.

In what follows we let $V_n = \{i \in \mathbf{Z}^d: i_j = 0, 1, ..., n-1, \ \forall j = 1, 2, ..., d\}$. Given $\omega \in \Omega$ we define its *n-periodic version* ω^n as follows:

$$\begin{aligned} \omega^n_t(i) &= \omega_t(i) \qquad \text{for} \quad 0 \leqslant t \leqslant n, \quad i \in V_n \\ \omega^n_{t+hn}(i+kn) &= \omega^n_t(i) \qquad \text{for} \quad h \in \mathbf{Z}, \quad k \in \mathbf{Z}^d \end{aligned}$$

where

$$kn = (k_1 n, ..., k_d n)$$

For $\Gamma \subset \mathbf{R} \times \mathbf{Z}^d$ we let $\mathscr{F}_\Gamma$ be the σ-field of subsets of Ω generated by the projections $\{\pi_{t,i}: (t, i) \in \Gamma\}$, where $\pi_{t,i}(\omega) = \omega_t(i)$. Sometimes we will use for $\mathscr{F}_\Gamma$ the notation $\sigma\{w_t(i): (t, i) \in \Gamma\}$. In the following definition we denote by $\mathscr{B}(\Omega)$ the set of bounded measurable functions $\Omega \to \mathbf{R}$.

Definition 2.2. Let $\omega \in \Omega$ and $\phi \in \mathscr{B}(\Omega)$. The nth empirical process $R_{n,\omega}$ is the element of $\mathcal{M}_s(\Omega)$ whose expectations are defined as follows:

$$E^{R_{n,\omega}}(\phi) = \frac{1}{n^{d+1}} \sum_{i \in V_n} \int_0^n \phi(\theta_{s,i}\omega^n)\, ds \tag{2}$$

Notice that, in order to make $R_{n,\omega}$ stationary, it is essential to use the nth-periodic version of ω in (2). We also remark that the map

$$\begin{aligned} &\Omega \to \mathcal{M}_s(\Omega) \\ &\omega \mapsto R_{n,\omega} \end{aligned}$$

is $\mathscr{F}_{[0,n] \times V_n}$-measurable. In the rest of the paper the σ-field $\mathscr{F}_{[0,n] \times V_n}$ will be simply denoted by $\mathscr{F}_n$.

Some more notations are now needed. We introduce on $\mathbf{Z}^d$ the lexicographic total order, and denote by $\prec$ the corresponding order relation. Consider the set

$$\Gamma^- = \{(t, i) \in \mathbf{R} \times \mathbf{Z}^d: t \leqslant 1, i \prec 0 \text{ or } t \leqslant 0, i \in \mathbf{Z}^d\}$$

and let $\mathscr{F}^- = \mathscr{F}_{\Gamma^-}$. For $Q \in \mathscr{M}_s(\Omega)$ we let Q_ω denote the regular conditional probability distribution (r.c.p.d.) of Q with respect to $\mathscr{F}^-$. In the following definition $dQ_\omega/dP_{0,\omega_0} | \mathscr{F}_1$ denotes the Radon–Nikodym derivative of the indicated measures restricted to the σ-field $\mathscr{F}_1$.

Definition 2.3. Let $Q \in \mathscr{M}_s(\Omega)$. The relative entropy of Q with respect to the Markov family $\{P_{0,\xi}: \xi \in \{-1, 1\}^{\mathbf{Z}^d}\}$ is defined by

$$H(Q) = E^Q \left\{ \log \left(\frac{dQ_\omega}{dP_{0,\omega_0}} \bigg| \mathscr{F}_1 \right) \right\} \tag{3}$$

where $H(Q) = +\infty$ if the Radon–Nikodym derivative in (3) is not defined or its logarithm is not in $L^1(Q)$.

In what follows, for $\omega \in \Omega$, $i \in \mathbf{Z}^d$, $t \in \mathbf{R}$, we let

$$N_t(i) = \sum_{0 \leqslant s \leqslant t} \frac{|\omega_{s^-}(i) - \omega_s(i)|}{2} + \delta_{1,\omega_0(i)}$$

Notice that $\omega_t(i) = (-1)^{N_t(i)}$. Now suppose we fix $\omega' \in \Omega$. For an arbitrary $\omega \in \Omega$ we define

$$\omega_t^{n,\omega'}(i) = \begin{cases} \omega_t(i) & \text{if } \; i \in V_n \\ \omega'_t(i) & \text{otherwise} \end{cases}$$

In what follows we briefly write $c^{\omega'}(\omega_t)$ in place of $c(\omega_t^{n,\omega'})$. The missing index n will be clear from the context. We then consider the following expression:

$$Z_{n,\omega'}(\omega) = \exp\left[\sum_{i \in V_n} \left\{ \int_0^n [1 - c^{\omega'}(\theta_i \omega_t^n)] \, dt + \int_0^n \log c^{\omega'}(\theta_i \omega_{t^-}) \, dN_t(i) \right\} \right]$$

By the Girsanov formula for point processes[9] it follows that, for every ω',

$$E^{P_{0,\xi}}\{Z_{n,\omega'}\} = 1$$

and the process whose law on $D([0, n], \{-1, 1\}^{V_n})$ is $P^c_{n,\omega'}$ defined by

$$dP^c_{n,\omega'}/dP_{0,\omega'_0} = Z_{n,\omega'}$$

is a spin process (on the finite lattice V_n) with flip rates $c^{\omega'}(\theta_i \omega)$, $i \in \mathbf{Z}^d$.

Notice that we can repeat the same argument when ω' is not fixed, but is a function of ω. In other words, denoting by $\mathscr{B}_n$ the set of all measurable functions

$$D([0,n],\{-1,1\}^{V_n}) \to D([0,n],\{-1,1\}^{V_n^c})$$

for $\omega' \in \mathscr{B}_n$ we can define $P^c_{n,\omega'}$ by

$$\frac{dP^c_{n,\omega'}}{dP_{0,\omega'_0}}(\omega) = Z_{n,\omega'(\omega)}(\omega)$$

The main result of ref. 4 is the following large-deviation principle.

Theorem 1. Let A be a Borel measurable subset of $\mathscr{M}_s(\Omega)$, and denote by $\mathring{A}$ and $\bar{A}$ its interior and its closure, respectively. Then

$$\begin{aligned}
-\inf_{Q\in \mathring{A}} H^c(Q) &\leqslant \liminf_{n\to\infty} \frac{1}{n^{d+1}} \inf_{\omega'\in\mathscr{B}_n} \log P^c_{n,\omega'}\{R_{n,\omega}\in A\} \\
&\leqslant \liminf_{n\to\infty} \frac{1}{n^{d+1}} \inf_{\xi\in\{-1,1\}^{\mathbb{Z}^d}} \log P^c_{0,\xi}\{R_{n,\omega}\in A\} \\
&\leqslant \limsup_{n\to\infty} \frac{1}{n^{d+1}} \sup_{\xi\in\{-1,1\}^{\mathbb{Z}^d}} \log P^c_{0,\xi}\{R_{n,\omega}\in A\} \\
&\leqslant \limsup_{n\to\infty} \frac{1}{n^{d+1}} \sup_{\omega'\in\mathscr{B}_n} \log P^c_{n,\omega'}\{R_{n,\omega}\in A\} \\
&\leqslant -\inf_{Q\in\bar{A}} H^c(Q)
\end{aligned}$$

where

$$H^c(Q) = H(Q) - E^Q\left[1 - c(\omega_0) - \int_0^1 \log c(\omega_{t^-})\, dN_t(0)\right]$$

In a large-deviation principle it is particularly relevant to determine the zeros of the rate function $H^c(\cdot)$. The following is the main result contained in ref. 5. We need to use the σ-field $\mathscr{F}^p = \sigma\{\omega_t(i)\colon t\leqslant 0\}$ and, for a given $Q\in\mathscr{M}_s(\Omega)$, we let Q^p_ω denote its r.c.p.d. with respect to $\mathscr{F}^p$.

Theorem 2. For every $Q\in\mathscr{M}_s(\Omega)$ we have $H^c(Q)\geqslant 0$ and $H^c(Q)=0$ if and only if $Q^p_\omega = P^c_{0,\omega_0}$, Q-a.s. In other words $H^c(Q)=0$ if and only if Q is a stationary Markovian measure generated by L^c.

The following proposition, also proved in ref. 5, establishes some properties of the rate function $H^c(\cdot)$ that will be used later in this paper.

Theorem 3. The rate function $H^c\colon \mathcal{M}_s(\Omega) \to \mathbf{R}^+$ is lower semicontinuous and has compact level sets, i.e., for every $l \geqslant 0$ the set $\{Q \in \mathcal{M}_s(\Omega)\colon H^c(Q) \leqslant l\}$ is compact in the weak topology.

We conclude this section by stating a technical result, proved in ref. 5, that will be very useful later.

Lemma 2.4. For every $n \in \mathbf{N}$ the following inequalities hold:

$$\inf\{Z_{n,\omega'}(\omega)\colon \omega' \in \mathscr{B}_n\} \leqslant \frac{dP^c_{0,\omega_0}}{dP_{0,\omega_0}}\bigg|_{\mathscr{F}_n}$$

$$\leqslant \sup\{Z_{n,\omega'}(\omega)\colon \omega' \in \mathscr{B}_n\}$$

3. CRITICAL LARGE DEVIATIONS

By the results in Section 2 we know that the sequence of random measures $\{R_{n,\omega}\}_{n \in \mathbf{N}}$ approaches with probability 1, as $n \to \infty$, the set $L_0 = \{Q \in \mathcal{M}_s(\Omega)\colon H^c(Q) = 0\}$. If the system is ergodic, then, by Theorem 2, L_0 has a unique element Q^*, and $R_{n,\omega}$ converges weakly to Q^* with probability 1. In the nonergodic case L_0 has more than one element, so that $\{R_{n,\omega}\}$ may have more than one limit point. A more detailed knowledge of the behavior of $R_{n,\omega}$ for large n is needed in this case. In particular we want to show that the probability that $R_{n,\omega}$ is close to any element of L_0 does not decrease too fast in n. This is the most crucial fact in our ergodicity test. The leading idea in our argument is rather standard (e.g., ref. 7), but we could not avoid a number of nontrivial technicalities.

Let $\mathscr{G}_n$ be the σ-field $\sigma\{\omega_t(i)\colon i \notin V_n,\ t \in [0, n]\}$, and denote by $P_{\mathscr{G}_n,\omega'}$ the r.c.p.d. of P^c_{0,ω'_0} with respect to $\mathscr{G}_n$ restricted to $\mathscr{F}_n$, which is a measurable function of $\omega' \in D([0, n], \{-1, 1\}^{V_n^c})$ taking value in the space of probability measures on $\mathscr{F}_n$. Moreover, we denote by r an integer such that $c(\sigma)$ depends only on $(\sigma(i)\colon i \in V_r)$.

Lemma 3.1. For $\lambda > 0$ let P^λ_{0,ω_0} be the (noninteracting) spin system with all flip rates equal to λ. Then

$$\frac{dP^c_{\mathscr{G}_n,\omega'}}{dP^\lambda_{0,\omega'_0}}\bigg|_{\mathscr{F}_n}(\omega) = Z^{-1}_{\lambda,n}(\omega') \exp\left[\sum_{i \in V_n + V_r} \left\{\int_0^n [\lambda - c^{\omega'}(\theta_i\omega_t)]\,dt + \int_0^n \log\frac{c^{\omega'}(\theta_i\omega_{t^-})}{\lambda}\,dN_t(i)\right\}\right]$$

where $Z^{-1}_{\lambda,n}$ is a normalization factor.

Proof. Let R be the probability measure on Ω defined by

$$\frac{dP^c_{0,\omega'_0}}{dR} = \exp\left[\sum_{i \in V_n + V_r} \left\{\int_0^n [\lambda - c^{\omega'}(\theta_i \omega_t)]\, dt + \int_0^n \log \frac{c^{\omega}(\theta_i \omega_{t^-})}{\lambda}\, dN_t(i)\right\}\right]$$

By the Girsanov Theorem, R is a spin-flip system with flip rates

$$\tilde{c}(i, \sigma) = \begin{cases} \lambda & \text{if} \quad i \in V_n + V_r \\ c(\theta_i \sigma) & \text{otherwise} \end{cases}$$

Notice that $R = P^{\lambda}_{0,\omega'_0}$ in $\mathscr{F}_n$ and that, under R, $\mathscr{F}_n$ and $\mathscr{G}_n$ are stochastically independent. The conclusion now easily follows. ∎

In what follows m and M denote $\min_\sigma c(\sigma)$ and $\max_\sigma c(\sigma)$, respectively, and, for $\Lambda \subset \mathbf{Z}^d$,

$$\partial(\Lambda) = \{i \in \Lambda : i + V_r \not\subset \Lambda\}$$

Moreover, we let $\nu_n(\Lambda, \omega)$ be the number of jumps from time $t = 0$ to $t = n$ of the components in Λ of the path ω.

Lemma 3.2. Let $A_n \in \mathscr{F}_n$, $C_1 = 4drM$, and $C_2 = \log(M/m)$. Then, for every $\omega' \in \Omega$ we have

$$P^c_{n,\omega'}(A_n) \leqslant P^c_{\mathscr{G}_n,\omega'}(A_n) \exp[C_1 n^d + C_2 \nu_n(\partial(V_n + V_r), \omega')]$$

Proof. By Lemma 3.1, for every $\lambda > 0$ we have

$$\begin{aligned} \frac{dP^c_{n,\omega'}}{dP^c_{\mathscr{G}_n,\omega'}} &= \frac{dP^c_{n,\omega'}}{dP^{\lambda}_{0,\omega'_0}} \frac{dP^{\lambda}_{0,\omega'_0}}{dP^c_{\mathscr{G}_n,\omega'}} \\ &= Z_{n,\lambda}(\omega') \exp\left[-\sum_{i \in \partial(V_n + V_r)} \left\{\int_0^n [\lambda - c^{\omega'}(\theta_i \omega_t)]\, dt \right.\right. \\ &\qquad \left.\left. + \int_0^n \log \frac{c^{\omega'}(\theta_i \omega_{t^-})}{\lambda}\, dN_t(i)\right\}\right] \end{aligned}$$

Now we let $\lambda = M$ and we estimate $Z_{n,M}$:

$$\begin{aligned}
Z_{n,M}(\omega') &= E^{P^M_{0,\omega'_0}|\mathscr{F}_n}\Bigg[\exp\Bigg(\sum_{i\in V_n+V_r}\Bigg\{\int_0^n [M-c^{\omega'}(\theta_i\omega_t)]\,dt \\
&\qquad + \int_0^n \log\frac{c^{\omega'}(\theta_i\omega_{t}-)}{M}\,dN_t(i)\Bigg\}\Bigg)\Bigg] \\
&\leqslant [\exp(4drMn^d)]\,E^{P^M_{0,\omega'_0}|\mathscr{F}_n}\exp\Bigg(\sum_{i\in V_n}\Bigg\{\int_0^n [\lambda-c^{\omega'}(\theta_i\omega_t)]\,dt \\
&\qquad + \int_0^n \log\frac{c^{\omega'}(\theta_i\omega_{t}-)}{\lambda}\,dN_t(i)\Bigg\}\Bigg) = \exp(4drMn^d)
\end{aligned}$$

where the last equality follows from the Girsanov Theorem. Thus

$$\frac{dP^c_{n,\omega'}}{dP^c_{\mathscr{G}_n,\omega'}} \leqslant \exp(4drMn^d)\exp\left[\left(\log\frac{M}{m}\right)\nu_n(\partial(V_n+V_r),\omega')\right]$$

and the conclusion follows. ∎

The next is a technical lemma that will be used several times later.

Lemma 3.3. Let $f_1, f_2,\ldots, f_k$ be increasing functions $\mathbf{N}\to\mathbf{R}$, and $A_1, A_2,\ldots, A_k$ be finite subsets of $\mathbf{Z}^d$. Moreover, for $i=1,2$, let $c_i(\sigma)$ be positive local functions such that $c_1(\sigma)\leqslant c_2(\eta)$ for every σ,η. Then

$$E^{P^{c_1}_{0,\xi}}\left\{\prod_i f_i(\nu_1(A_i,\omega))\right\} \leqslant E^{P^{c_2}_{0,\xi}}\left\{\prod_i f_i(\nu_1(A_i,\omega))\right\}$$

Proof. We just sketch the proof, since it comes from very standard arguments. The main idea is to construct the *basic coupling* of the two spin systems, i.e., a probability measure on $\Omega\times\Omega$ whose marginals are $P^{c_1}_{0,\xi}$ and $P^{c_2}_{0,\xi}$. The basic coupling is described in ref. 8, Chapter 2. The measure on $\Omega\times\Omega$ so constructed gives probability one to the pairs (ω,ω') having the following property: for every $t\in\mathbf{R}$ and $i\in\mathbf{Z}^d$, if $\omega_t(i)=\omega'_t(i)$, then the probability that, after time t, $\omega(i)$ flips before $\omega'(i)$ is zero. Taking into account that $\omega_0=\xi=\omega'_0$, it follows that, in the time interval $[0,1]$, $\omega(i)$ makes less flips that $\omega'(i)$ for any $i\in\mathbf{Z}^d$. The statement of the lemma easily follows. ∎

Lemma 3.4. Let $Q\in\mathscr{M}_s(\Omega)$, and let A be an open neighborhood of Q. Then there exists another open neighborhood B of Q such that, for every $\omega',\omega''\in\mathscr{B}_n$ and for n large enough,

$$P^c_{n,\omega'}\{R_{n,\omega}\in A\} \geqslant [P^c_{n,\omega''}\{R_{n,\omega}\in B\}]^2\,e^{-C_3n^d}$$

where

$$C_3 = 2drM\left(2+\frac{M^2}{m^2}\right) + M + \log\frac{2}{1-e^{-2m}}$$

Proof. We first consider the case $\omega_0' = \omega_0''$. We have

$$\frac{dP^c_{n,\omega''}}{dP^c_{n,\omega'}}(\omega) = \exp\left\{\sum_{i\in\partial(V_n)}\left[\int_0^n (c^{\omega'} - c^{\omega''})\,dt + \int_0^n \log\frac{c^{\omega''}}{c^{\omega'}}\,dN_t(i)\right]\right\}$$
$$\leqslant \exp(2drMn^d)\exp\left[\log\frac{M}{m}\,v_n(\partial(V_n),\omega)\right]$$

Thus, using the Schwartz inequality and Lemma 3.3,

$$\begin{aligned}P^c_{n,\omega''}\{R_{n,\omega}\in A\} &\leqslant [\exp(2drMn^d)]\\ &\quad\times E^{P^c_{n,\omega'}}\left\{\chi_{\{R_{n,\omega}\in A\}}\exp\left[\log\frac{M}{m}\,v_n(\partial(V_n),\omega)\right]\right\}\\ &\leqslant [\exp(2drMn^d)][P^c_{n,\omega'}\{R_{n,\omega}\in A\}]^{1/2}\\ &\quad\times\left(E^{P^c_{n,\omega'}}\left\{\exp\left[\log\frac{M^2}{m^2}\,v_n(\partial(V_n)\omega)\right]\right\}\right)^{1/2}\\ &\leqslant [\exp(2drMn^d)][P^c_{n,\omega'}\{R_{n,\omega}\in A\}]^{1/2}\exp\left(M\,dr\,\frac{M^2}{m^2}\,n^d\right)\\ &= [P^c_{n,\omega'}\{R_{n,\omega}\in A\}]^{1/2}\exp(D_1 n^d)\end{aligned}$$

with $D_1 = drM(2 + M^2/m^2)$. Now we consider ω', ω'' to be arbitrary elements of $\mathscr{B}_n$. In particular we let $\xi' = \omega_0'$, $\xi'' = \omega_0''$. To emphasize the dependence on the initial condition, for the rest of the proof we write $P^c_{n,\omega',\xi'}$ rather than just $P^c_{n,\omega'}$. We define $\tilde{R}_{n,\omega}\in\mathscr{M}_s(\Omega)$ by

$$\tilde{R}_{n,\omega} = R_{n-1,\theta_{1,0}\omega}$$

It is clear that there is an open set $\tilde{A}\subset\mathscr{M}_s(\Omega)$ containing Q such that if $\tilde{R}_{n,\omega}\in\tilde{A}$, then $R_{n,\omega}\in A$. We have

$$\begin{aligned}P^c_{n,\omega',\xi'}\{R_{n,\omega}\in A\} &\geqslant P^c_{n,\omega',\xi'}\{\tilde{R}_{n,\omega}\in\tilde{A}\}\\ &= \int P^c_{n,\theta_{1,0}\omega',\eta_1}\{\tilde{R}\in\tilde{A}\}\;P^c_{n,\omega',\xi'}(d\eta)\end{aligned}$$

Now let

$$\phi(\eta_1) = P^c_{n,\theta_{1,0}\omega',\eta_1}\{\tilde{R}_{n,\omega}\in\tilde{A}\}$$

For $\lambda > 0$, it is easy to show that

$$\inf_{\sigma} P^{\lambda}_{0,\xi'}\{\eta_1 = \sigma \text{ on } V_n\} = \left(\frac{1-e^{-2\lambda}}{2}\right)^{n^d}$$

Therefore

$$\begin{aligned} &P^c_{n,\omega',\xi'}\{R_{n,\omega} \in A\} \\ &\quad \geqslant \int \phi(\eta_1) \exp\left\{\sum_{i \in V_n}\left[\int_0^1 (m - c^{\omega'})\,dt + \int_0^1 \log\frac{c^{\omega'}}{m}\,dN_t(i)\right]\right\} P^m_{0,\xi'}(d\eta) \\ &\quad \geqslant [\exp(-Mn^d)]\left(\frac{1-\exp(-2m)}{2}\right)^{n^d} \phi(\xi'') \\ &\quad = [\exp(-D_2 n^d)]\,\phi(\xi'') \end{aligned}$$

with

$$D_2 = M + \log\frac{2}{1-e^{-2m}}$$

On the other hand, by what we have seen in the first part of the proof

$$\begin{aligned} \phi(\xi'') &\geqslant e^{-2D_1 n^d}[P^c_{n,\omega''}\{\tilde{R}_{n,\omega} \in \tilde{A}\}]^2 \\ &\geqslant e^{-2D_1 n^d}[P^c_{n,\omega''}\{R_{n,\omega} \in B\}]^2 \end{aligned}$$

for n large, and for B a suitable open set containing Q. ∎

The following result gives a lower bound for the large deviations in special subsets of L_0.

Theorem 4. Let $A \subset \mathcal{M}_s(\Omega)$ be an open set such that there exists $Q \in L_0 \cap A$ with Q ergodic. Then

$$\inf_{\xi \in \{-1,1\}^{\mathbb{Z}^d}} P^c_{0,\xi}\{R_{n,\omega} \in A\} \geqslant \inf_{\omega' \in \mathscr{B}_n} P^c_{n,\omega'}\{R_{n,\omega} \in A\} \geqslant e^{-C_4 n^d}$$

for n large enough and $C_4 = C_3 + 1$.

Proof. The first inequality comes from Lemma 2.4. Let $\omega' \in \mathscr{B}_n$. By Lemma 3.4 there is B open, $Q \in B$, such that

$$\begin{aligned} P^c_{n,\omega'}\{R_{n,\omega} \in A\} &\geqslant e^{-C_3 n^d}[\sup_{\omega'' \in \mathscr{B}_n} P^c_{n,\omega''}\{R_{n,\omega} \in B\}]^2 \\ &\geqslant e^{-C_3 n^d}[E^Q\{P^c_{0,\omega_0}\{R_{n,\omega} \in B\}\}]^2 \\ &= e^{-C_3 n^d}[Q\{R_{n,\omega} \in B\}]^2 \geqslant e^{-C_4 n^d} \end{aligned}$$

for n large enough, where we have used again Lemma 2.4, the fact that Q is ergodic, and that, by Theorem 2, $Q^P_\omega = P^c_{0,\omega_0}$. ∎

Corollary 3.5. Let A be as in Theorem 4. Then for every $\omega' \in \Omega$ and n sufficiently large

$$P^c_{\mathscr{G}_n,\omega'}\{R_{n,\omega} \in A\} \geqslant e^{-Cn^d} e^{-C_2 v_n(\partial(V_n+V_r),\omega')} \tag{4}$$

Proof. It follows from Theorem 4 and Lemma 3.2. ∎

Remark 3.6. The constants C, C_2 that make (4) true are of course not optimal, i.e., one could find smaller values for them keeping (4) true. All we wanted to show is that one can determine constants that are simple explicit functions of M, m, d, r. The estimate (4) is the key fact in the ergodicity test we develop in the next section.

4. THE ERGODICITY TEST

In this section we develop a test to determine whether the spin system with generator L^c has a unique translation-invariant stationary measure. By Theorem 2 this is equivalent to determining whether L_0 has a unique element.

The method shown here consists in computing *moving averages* of a single realization. For $l \leqslant n$ we consider the set of empirical measures

$$\Delta_{n,l}(\omega) = \{R_{l,\theta_{\tau,i}\omega} : (\tau, i) \in \mathbf{Z} \times \mathbf{Z}^d, (\tau, i) + ([0, l] \times V_l) \subset [0, n] \times V_n\}$$

Notice that $\Delta_{n,l}(\omega) \subset \mathscr{M}_s(\Omega)$. We want to show that, if $l = l(n)$ depends on n in a suitable way, then the convex hull of $\Delta_{n,l(n)}(\omega)$, denoted by $\mathscr{C}(\Delta_{n,l(n)}(\omega))$, approaches L_0. To be more precise, let $\mathbf{d}(\cdot,\cdot)$ be the Prohorov metric on $\mathscr{M}_s(\Omega)$ and, for $A, B \subset \mathscr{M}_s(\Omega)$, let

$$\mathbf{d}(A, B) = \max\{\sup_{P \in A} \mathbf{d}(P, B), \sup_{Q \in B} \mathbf{d}(Q, A)\}$$

where $\mathbf{d}(P, B) = \inf_{Q \in B} \mathbf{d}(P, Q)$. We want to find conditions on $l(n)$ for which, for any $\delta > 0$,

$$\sup_{\xi} P^c_{0,\xi}\{\omega : \mathbf{d}(\mathscr{C}(\Delta_{n,l(n)}(\omega)), L_0) > \delta\} = f(n, \delta)$$

goes to zero as $n \to \infty$, and we also determine an asymptotic upper bound for $f(n, \delta)$. The proof is divided into two propositions.

Proposition 4.1. Let $l(n)$ be such that

$$\lim_{n\to\infty}\frac{\log n}{l^{d+1}(n)}=0$$

Then for every $\delta>0$ there exists $k(\delta)>0$ such that

$$P^c_{0,\xi}\{\omega: \max_{R\in\Delta_{n,l(n)}(\omega)}\mathbf{d}(R,L_0)>\delta\}\leqslant\exp[-k(\delta)\,l^{d+1}(n)]$$

Proof. Define

$$A_{n,\delta}=\{\omega: \max_{R\in\Delta_{n,l(n)}(\omega)}\mathbf{d}(R,L_0)>\delta\}$$

and

$$2k(\delta)=\inf\{H^c(Q): \mathbf{d}(Q,L_0)>\delta\}$$

Since H^c is l.s.c. and has compact level sets, we have $k(\delta)>0$. By the uniform upper bound for the large deviations (Theorem 1) there exists a sequence α_l such that $\alpha_l\to 0$ as $l\to\infty$ and

$$\begin{aligned}P^c_{0,\xi}\{\mathbf{d}(R_{l,\theta_{\tau,i}\omega},L_0)>\delta\}&\leqslant\exp\{-l^{d+1}[2k(\delta)-\alpha_l]\}\\&\leqslant\exp[-\tfrac{3}{2}k(\delta)\,l^{d+1}]\end{aligned}$$

for l sufficiently large. Therefore, for n large enough we have

$$\begin{aligned}P^c_{0,\xi}(A_{n,\delta})&\leqslant\sum_{\tau,i}P^c_{0,\xi}\{\mathbf{d}(R_{l(n),\theta_{\tau,i}\omega},L_0)>\delta\}\\&\leqslant n^{d+1}\exp\left[-\frac{3}{2}k(\delta)\,l^{d+1}(n)\right]\\&=\exp\left\{-l^{d+1}(n)\left[\frac{3}{2}k(\delta)-\frac{\log n^d}{l^{d+1}(n)}\right]\right\}\\&\leqslant\exp[-k(\delta)\,l^{d+1}(n)]\end{aligned}$$

where the sum in the first line is over the pairs $(\tau,i)\in\mathbf{Z}\times\mathbf{Z}^d$ for which $(\tau,i)+([0,l(n)]\times V_{l(n)})\subset[0,n]\times V_n$. ∎

Remark 4.2. The meaning of Proposition 4.1 is that for the empirical measures in $\Delta_{n,l(n)}$ not to fall outside L_0, $l(n)$ must be sufficiently large, namely larger than $\sim\log^{1/(d+1)}n$. If $l(n)$ is too small, then one is likely to observe "deviations" outside L_0. Indeed, the same argument of Proposition 4.1 (see ref. 3, Theorem 3.1) allows one to show that if $\liminf_n (\log n)/l^{d+1}(n) \geqslant t \geqslant 0$, then

$$\lim_{n \to \infty} P^c_{0,\xi}\{ \max_{R \in \Delta_{n,l(n)}} \mathbf{d}(R, L_t) > \delta\} = 0$$

for every $\delta > 0$, where $L_t = \{Q\colon H^c(Q) \leqslant t\}$.

Proposition 4.3. Let $l(n)$ be such that

$$\liminf \frac{\log(n^{d+1})}{l^d(n)} \geqslant D + \varepsilon$$

for some $\varepsilon > 0$, and for $D = C + 4rdM(1 - e^{-C_2})$, where r, M, C, C_2 are the constants introduced in Section 3. Then for every $\delta > 0$ and n sufficiently large we have

$$P^c_{0,\xi}\{\omega\colon \max_{Q \in L_0} \mathbf{d}(Q, \mathscr{C}(\Delta_{n,l(n)}(\omega))) > \delta\} < \exp\left[-\frac{e^{\varepsilon l^d(n)}}{l^{d+1}(n)}\right]$$

Proof. First we observe that, being compact, L_0 can be covered by a finite number of balls of radius $\delta/2$. It is therefore enough to show that

$$P^c_{0,\xi}\left\{\mathbf{d}(Q, \mathscr{C}(\Delta_{n,l(n)})) > \frac{\delta}{2}\right\} < \exp\left[-\frac{e^{\varepsilon l^d(n)}}{l^{d+1}(n)}\right]$$

for every $Q \in L_0$. Now let us denote by L_0^e the set of ergodic measures in L_0; by the Ergodic Decomposition Theorem L_0 is the convex hull of L_0^e. Thus, using the fact that for P, P', Q, $Q' \in \mathscr{M}_s(\Omega)$ and $0 \leqslant \alpha \leqslant 1$

$$\mathbf{d}(\alpha P + (1-\alpha)\,P', \alpha Q + (1-\alpha)\,Q') \leqslant \max[\mathbf{d}(P, Q), \mathbf{d}(P', Q')]$$

all we have to show is that, for every Q in L_0^e,

$$P^c_{0,\xi}\left\{\mathbf{d}(Q, \Delta_{n,l(n)}) > \frac{\delta}{2}\right\} < \exp\left[-\frac{e^{\varepsilon l^d(n)}}{l^{d+1}(n)}\right]$$

Now let Θ_d be the sublattice of $\mathbf{Z}^{d+1}$ of those points whose coordinates are multiples of $l(n)$, and

$$C_n = \{(\tau, i) \in \Theta_d\colon (\tau, i) + [0, l(n)] \times V_{l(n)} \subset [0, n] \times V_n\}$$

Clearly

$$\{\omega\colon \mathbf{d}(Q, \Delta_{n,l(n)}) > \delta/2\} \subset \bigcap_{(\tau,i)\in C_n} \{\omega\colon \mathbf{d}(Q, R_{l(n),\theta_{\tau,i}\omega}) > \delta/2\}$$

Let $W_{n,i}$ be the square $i + V_{l(n)}$, and

$$W'_{n,i} = \{j \in W_{n,i}\colon j + V_r \subset W_{n,i}\}$$

Notice that $\{W'_{n,i}\colon i \in l(n)\,\mathbf{Z}^d\}$ are disjoint squares separated by corridors of size $2r$. We also denote by $R'_{l(n),\theta_{\tau,i}\omega}$ the empirical measure referred to the "rectangle" $[\tau, \tau + l(n)] \times W'_{n,i}$, with the usual periodic boundary conditions. For n large and for every Q we have

$$\begin{aligned}\{\mathbf{d}(Q, R_{l(n),\theta_{\tau,i}\omega}) > \delta/2\} &\supset \{\mathbf{d}(Q, R'_{l(n),\theta_{\tau,i}\omega}) > \delta/3\}\\ &\supset \{\mathbf{d}(Q, R_{l(n),\theta_{\tau,i}\omega}) > \delta/4\} \qquad (5)\end{aligned}$$

Thus

$$\begin{aligned}&P^c_{0,\xi}\{\mathbf{d}(Q, \Delta_{n,l(n)}) > \delta/2\}\\ &\quad\geqslant P^c_{0,\xi}\left[\bigcap_{(\tau,i)\in C_n} \{\mathbf{d}(Q, R'_{l(n),\theta_{\tau,i}\omega}) > \delta/3\}\right]\\ &\quad= E^{P^c_{0,\xi}}\left[\prod_{(\tau,i)\in C_n} P^c_{\mathscr{G}_{l(n),\theta_{\tau,i}\omega}}\{\mathbf{d}(Q, R'_{l(n),\omega'}) > \delta/3\}\right]\\ &\quad= E^{P^c_{0,\xi}}\left[\prod_{(\tau,i)\in C_n} (1 - P^c_{\mathscr{G}_{l(n),\theta_{\tau,i}\omega}}\{\mathbf{d}(Q, R_{l(n),\omega'}) \leqslant \delta/4\})\right] = (A)\end{aligned}$$

where we have used Lemma 3.1 and (5). By Corollary 3.5

$$(A) \leqslant E^{P^c_{0,\xi}}\left[\prod_{(\tau,i)\in C_n} (1 - e^{-Cl^d(n)} e^{-C_2 v_{l(n)}(\partial(V_{l(n)} + V_r), \theta_{\tau,i}\omega)})\right] = (B) \qquad (6)$$

Now let $C_{n,0}$ be the subset of C_n made of those pairs (τ, i) such that $\tau = 0$. By the Markov property and Lemma 3.3

$$\begin{aligned}(B) &\leqslant \left[\sup_\xi E^{P^c_{0,\xi}}\left[\prod_{(0,i)\in C_{n,0}} (1 - e^{-Cl^d(n)} e^{-C_2 v_{l(n)}(\partial(V_{l(n)} + V_r), \theta_{0,i}\omega)})\right]\right]^{|C_n|/|C_{n,0}|}\\ &\leqslant \left[\sup_\xi E^{P^M_{0,\xi}}\left[\prod_{(0,i)\in C_{n,0}} (1 - e^{-Cl^d(n)} e^{-C_2 v_{l(n)}(\partial(V_{l(n)} + V_r), \theta_{0,i}\omega)})\right]\right]^{|C_n|/|C_{n,0}|}\end{aligned}$$

On the other hand,

$$
\begin{aligned}
E^{P^M_{0,\xi}}&\left[\prod_{(0,i)\in C_{n,0}} (1-e^{-Cl^d(n)}e^{-C_2 v_{l(n)}(\partial(V_{l(n)}+V_r),\theta_{0,i}\omega)})\right] \\
&= \prod_{(0,i)\in C_{n,0}} E^{P^M_{0,\xi}}(1-e^{-Cl^d(n)}e^{C_2 v_{l(n)}(\partial(V_{l(n)}+V_r),\theta_{0,i}\omega)}) \\
&= \prod_{(0,i)\in C_{n,0}} [1-e^{-Cl^d(n)}E^{P^M_{0,\xi}}(e^{C_2 v_{l(n)}(\partial(V_{l(n)}+V_r),\theta_{0,i}\omega)})] \\
&= [1-e^{-Dl^d(n)}]^{|C_{n,0}|} \\
&\leqslant \exp[-e^{-Dl^d(n)}\,|C_{n,0}|]
\end{aligned}
$$

Thus

$$
\begin{aligned}
P^c_{0,\xi}&\{\omega\colon \mathbf{d}(Q, \Delta_{n,l(n)})>\delta/2\} \\
&\leqslant \exp[-|C_n|\,e^{-Dl^d(n)}] \\
&\leqslant \exp\left\{-\frac{1}{l^{d+1}(n)}\exp[\log n^{d+1}-Dl^d(n)]\right\} \\
&\leqslant \exp\left[-\frac{e^{\varepsilon l^d(n)}}{l^{d+1}(n)}\right] \quad \blacksquare
\end{aligned}
$$

Propositions 4.1 and 4.3 yield the main result of this paper.

Theorem 5. Let $l(n)$ be such that

$$
\lim_{n\to\infty}\frac{(d+1)\log n}{l^d(n)} = D+\varepsilon \tag{7}
$$

for some $\varepsilon>0$. Then, for n large enough,

$$
\sup_{\xi\in\{-1,1\}^{\mathbb{Z}^d}} P^c_{0,\xi}\{\omega\colon \mathbf{d}(\mathscr{C}(\Delta_{n,l(n)}(\omega)), L_0)>\delta\} \leqslant 2e^{-k(\delta)l^{d+1}(n)}
$$

where $k(\delta)$ is the constant introduced in Proposition 4.1.

Proof. Just observe that, for every $\varepsilon>0$, one can take n large enough so that

$$
\exp\left[-\frac{e^{\varepsilon l^d(n)}}{l^{d+1}(n)}\right] < \exp[-k(\delta)\,l^{d+1}(n)] \quad \blacksquare \tag{8}
$$

Remark 4.4. In Theorem 5 the condition on $l(n)$ is explicit, since the constant D is actually computable. The constant $k(\delta)$, however, is not computable in practice, although we know it is strictly positive.

Remark 4.5. When the system is reversible with respect to a Gibbs measure, one may ask how much information is gained using the whole path rather than sampling at a fixed large time t and using Comets' argument. The answer to this question would consist in analyzing the constant $k(\delta)$ appearing in Theorem 5 and comparing it with the corresponding constant in Comets' paper. Such an analysis requires a detailed knowledge of the rate functions, which may be available only in some special model.

Remark 4.6. A suitable version of Theorem 5 indeed holds even if we replace $\mathscr{C}(\Delta_{n,l(n)})$ by $\Delta_{n,l(n)}$ itself. The reason why we have considered the convex hull of $\Delta_{n,l(n)}$ is that we have proved the lower bound in Theorem 4 only for open sets A containing ergodic measures in L_0. Actually, it is possible to prove Theorem 4 for every open set A having nonempty intersection with L_0, and the price to pay is that the constant C depends on A.

We have chosen not to give such proofs for three reasons. The first is that, although the argument to extend Theorem 4 is standard (see e.g., ref. 6), the details in our model are rather technical. Second, in condition (7), the constant D would be an unknown function of δ, so the condition on $l(n)$ is no longer explicit. This could be fixed by choosing $l(n)$ such that

$$\lim_n \frac{\log n}{l^d(n)} = +\infty$$

$$\lim_n \frac{\log n}{l^{d+1}(n)} = 0$$

but in this way we would get a worse rate of convergence in Theorem 5. Finally, if we know that $\Delta_{n,l(n)}$ spreads out in a set of diameter $>r$ (or, conversely, concentrates in a set of diameter $<r$), then the same is true for its convex hull. The information on ergodicity we wanted to detect is therefore contained in $\Delta_{n,l(n)}$.

In particular the following result is an easy consequence of Theorems 5 and 2.

Corollary 4.7. Let $f: \{-1, 1\}^{\mathbf{Z}^d} \to \mathbf{R}$ be a continuous function, and $l(n)$ be as in Theorem 5. Define

$$I_n^f(\omega) = [a_n(\omega), b_n(\omega)]$$

where

$$a_n(\omega) = \inf\{E^R(f(\omega'_0)) \colon R \in \Delta_{n, l(n)}(\omega)\}$$
$$b_n(\omega) = \sup\{E^R(f(\omega'_0)) \colon R \in \Delta_{n, l(n)}(\omega)\}$$

and

$$I^f = [a, b]$$

with

$$a = \inf\{E^\mu(f) \colon \mu \text{ is an invariant measure for } L^c\}$$
$$b = \sup\{E^\mu(f) \colon \mu \text{ is an invariant measure for } L^c\}$$

Then, for every $\delta > 0$ there is $k > 0$, depending on δ and f, such that

$$\sup_\xi P^c_{0,\xi}\{\omega \colon \mathrm{dist}(I^f_n(\omega), I^f) > \delta\} \leqslant e^{-kl^{d+1}(n)}$$

where $\mathrm{dist}(\cdot, \cdot)$ denotes the usual distance between subsets of $\mathbf{R}$.

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